DE MOIVRE'S CENTRAL LIMIT THEOREM

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1. Introduction

In this basic form, the central limit theorem can be stated as follows:

Theorem 1.1 (Lindeberg–Lévy central limit theorem). *Suppose that $X_1, X_2, \ldots$ are i.i.d. mean 0 and variance 1 random variables, and let $a < b$ be fixed. Then,*

$$\mathbb{P}\{a\sqrt{n} \le X_1 + \cdots + X_n \le b\sqrt{n}\} \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

as $n \to \infty$.

In the case where X_j are symmetric ± 1 random variables $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$, the central limit theorem dates back to the French mathematician Abraham de Moivre (1667 - 1754).

Theorem 1.2 (de Moivre's central limit theorem). *Suppose that $X_1, \ldots, X_n$ are independent symmetric ± 1 random variables, and let $a < b$ be fixed. Then,*

$$\mathbb{P}\{a\sqrt{n} \le X_1 + \cdots + X_n \le b\sqrt{n}\} \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

as $n \to \infty$.

The purpose of this note is to give a short sketch of the proof of Theorem 1.2.

2. Proof of de Moivre's central limit theorem

2.1. **Ingredients.** Recall that the asymptotic series version of Stirling's formula implies that

$$x! = \sqrt{2\pi} x^{x+1/2} e^{-x} (1 + \mathcal{O}(1/x)), \quad \text{as} \quad x \to \infty, \tag{1}$$

and recall that the exponential and logarithm functions have asymptotic series

$$\log(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^2) \quad \text{and} \quad e^x = 1 + \mathcal{O}(x), \quad \text{as} \quad x \to 0. \tag{2}$$

2.2. **Sketch of proof.** For simplicity, assume that n is an even integer such that $X_1 + \ldots + X_n$ will always be even. Let $p_k := \mathbb{P}(X_1 + \cdots + X_n = 2k)$ such that

$$\mathbb{P}\{a\sqrt{n} \le X_1 + \cdots + X_n \le b\sqrt{n}\} = \sum_{a\sqrt{n}/2 \le k \le b\sqrt{n}/2} p_k$$

where the sum is over integers k between $a\sqrt{n}/2$ and $b\sqrt{n}/2$. In the following calculations, we use (1), (2), and the fact that $k = \mathcal{O}(\sqrt{n})$. As $n \to \infty$, we have

$$
\begin{aligned}
p_k &= 2^{-n} \frac{n!}{(n/2-k)!(n/2+k)!} \\
&\to 2^{-n} \frac{(2\pi)^{1/2} n^{n+1/2} e^{-n}}{2\pi (n/2-k)^{n/2-k+1/2} e^{-(n/2-k)} (n/2+k)^{n/2+k+1/2} e^{-(n/2+k)}} \\
&= \frac{2}{(2\pi n)^{1/2}} \left(1 - \frac{2k}{n}\right)^{-(n/2-k+1/2)} \left(1 + \frac{2k}{n}\right)^{-(n/2+k+1/2)} \\
&= \frac{2}{(2\pi n)^{1/2}} e^{-\left(\frac{n}{2}-k+\frac{1}{2}\right)\ln\left(1-\frac{2k}{n}\right) - \left(\frac{n}{2}+k+\frac{1}{2}\right)\ln\left(1+\frac{2k}{n}\right)} \\
&\to \frac{2}{(2\pi n)^{1/2}} e^{-\left(\frac{n}{2}-k+\frac{1}{2}\right)\left(-\frac{2k}{n}-\frac{2k^2}{n^2}\right) - \left(\frac{n}{2}+k+\frac{1}{2}\right)\left(\frac{2k}{n}-\frac{2k^2}{n^2}\right)} \\
&\to \frac{2}{(2\pi n)^{1/2}} e^{k - \frac{2k}{n} + \frac{k^2}{n} - k - \frac{2k^2}{n} + \frac{k^2}{n}} \\
&= \frac{2}{(2\pi n)^{1/2}} e^{-\frac{2k^2}{n}}.
\end{aligned}
$$

Thus, if $h := 2/\sqrt{n}$ we have

$$
\begin{aligned}
\sum_{a\sqrt{n}/2 \le k \le b\sqrt{n}/2} p_k &\to \sum_{a\sqrt{n}/2 \le k \le b\sqrt{n}/2} \frac{2}{(2\pi n)^{1/2}} e^{-\frac{2k^2}{n}} \\
&= \sum_{a \le hk \le b} \frac{1}{(2\pi)^{1/2}} e^{-\frac{(hk)^2}{2}} h, \\
&\to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,
\end{aligned}
$$

as $n \to \infty$, where the final limit follows from the fact that the second to last sum is a Reimann sum for the integral. This concludes the proof sketch.