# EULER-MACLAURIN

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## 1. Introduction

**1.1. Aysmptotic series.** Recall that we say that $f(h)$ has an asymptotic series

$$f(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \cdots \tag{1}$$

as $h \to 0$ if for any fixed integer $m \geq 0$

$$f(h) = a_0 + a_1 h + \cdots + a_m h^m + \mathcal{O}(h^{m+1}),$$

where the constant implicit in the big-$\mathcal{O}$ depends on $m$.

**1.2. Richardson extrapolation.** The technique of using tricks to cancel terms in asympotic series is called Richardson extrapolation. For example, if $f$ has asymptotic series (1) and we set

$$g(h) = 2f(h/2) - f(h),$$

then $g$ has asymptotic series

$$g(h) = a_0 + b_2 h^2 + b_3 h^3 + \cdots,$$

where $b_k = a_k(2^{-k+1} - 1)$.

**1.3. Taylor's Theorem.** If $f$ is a smooth real-valued function on $[a, b]$, then Taylor's Theorem says that $f(x + h)$ has an asymptotic series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

as $h \to 0$ assuming that $x, x + h \in [a, b]$. Indeed, the fact that $f(x + h)$ has this asymptotic series is justified by the Lagrange remainder formulation of Taylor's Theorem, which we can state as follows. Let $f$ be a real-valued function that is $n$ times continuously differentiable in $[a, b]$ and $(n + 1)$-times differentiable in $(a, b)$. Then, for all $x, x_0 \in [a, b]$

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1},$$

for some $\xi = \xi(x, x_0)$ in the open interval whose end points are $x, x_0$.

## 2. Euler-Maclaurin formula

2.1. **Trapezoid rule.** Suppose that $f$ is a smooth real-valued function on $[a,b]$, and let $h = (b-a)/n$ for a fixed integer $n$. The Trapezoid rule $T(h)$ with step length $h$ is defined by

$$T(h) = h\left(\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b)\right),$$

or put differently,

$$\frac{T(h)}{h} = \sum_{k=0}^{n} f(a+hk) - \frac{1}{2}(f(a)+f(b)).$$

2.2. **Basic statement.** Suppose that $f$ is a smooth real-valued function $[a,b]$. Informally speaking, the Euler-Macluarin formula says that trapezoid rule $T(h)$ has the asymptotic series which only has even powers of $h$ whose first few terms are

$$T(h) = \int_a^b f(x)dx + \frac{h^2}{12}\left(f'(b) - f'(a)\right) - \frac{h^4}{720}\left(f'''(b) - f'''(a)\right) + \frac{h^6}{30240}\left(f^{(5)}(b) - f^{(5)}(a)\right) + \cdots .$$

There are many ways to use the Euler-Maclaurin formula. For example, we could use the fact that $T(h)$ has an asymptotic series which only has even powers of $h$ to create an $\mathcal{O}(h^4)$ integration scheme by using Richardson Extrapolation

$$\frac{4T(h/2) - T(h)}{3} = \int_a^b f(x)dx + \mathcal{O}(h^4).$$

Alternatively, we could use the formula to perform an end point correction to Trapezoid rule by

$$T(h) - \frac{h^2}{12}(f'(b) - f'(a)) = \int_a^b f(x)dx + \mathcal{O}(h^4).$$

Both of these methods could be used to create higher order schemes for estimating the integral of $f$ over $[a,b]$.

2.3. **Precise statement with remainder formula.** Let $f$ be a real-valued function that $2r$ times continuously differentiable on $(a,b)$. Fix an integer $n \geq 1$ and let $h = (b-a)/n$. Then, the Euler-Macluarin formula states that[1]

$$\sum_{k=0}^{n} f(a+kh) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}(f(b)+f(a)) + \sum_{k=1}^{r-1} \frac{h^{2k-1}}{(2k)!} B_{2k}(f^{(2k-1)}(b) - f^{(2k-1)}(a)) + \frac{h^{2r}}{(2r)!} B_{2r} \sum_{k=0}^{n-1} f^{(2r)}(a+kh+\xi h),$$

for some $0 < \xi < 1$, where $B_k$ denotes the $k$-th Bernouli number, which have the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

[1] see for example page 806 of Abramowitz and Stegun

The first few Bernouli numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}.$$