GAUSSIAN QUADRATURE

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1. Introduction

1.1. **Legendre Polynomials.** Let $P_k(x)$ be the degree k Legendre polynomial. The Legendre polynomials are an orthogonal family of polynomials on $[-1,1]$ that obey the orthogonality relation

$$\int_{-1}^{1} P_j(x) P_k(x) dx = 0,$$

for $j \neq k$. The first few Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad \text{and} \qquad P_2(x) = \frac{3x^2 - 1}{2}.$$

The Legendre polynomials obey the recurrence relation

$$(k+1) P_{k+1}(x) = (2k+1) x P_k(x) - k P_{k-1}(x),$$

the differential equation

$$(1 - x^2) P_k''(x) - 2x P_k'(x) + k(k+1) P_k(x) = 0,$$

and Rodrigues' formula

$$P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} \left((x^2 - 1)^k \right),$$

for $k \geq 1$.

1.2. **Roots.** The k-th degree Legendre polynomial P_k has k distinct simple roots in the interval $[-1,1]$. Indeed, suppose that $x_1, \ldots, x_m$ are the points at which $P_k(x)$ changes sign in $[-1,1]$. Then, the integral

$$I = \int_{-1}^{1} P_k(x) \prod_{j=1}^{m} (x - x_j) dx$$

is non-zero since the integrand does not change sign. Since P_k is orthogonal to all polynomials of degree less than k we conclude that $m = k$.

1.3. **Weights.** Let $x_1, \ldots, x_n$ be the roots of $P_n(x)$. Consider the linear system of n equations and n unknowns $w_1, \ldots, w_n$

$$\sum_{k=1}^{n} P_j(x_k) w_k = \int_{-1}^{1} P_j(x) dx, \tag{1}$$

for $j = 0, \ldots, n-1$. The general form of the adjoint linear system of equations is

$$\sum_{j=0}^{n-1} a_j P_j(x_k) = y_k \tag{2}$$

for $k = 1, \ldots, n$, which determines a polynomial $p(x)$ of degree $n-1$

$$p(x) = \sum_{j=0}^{n-1} a_j P_j(x),$$

that interpolates the values $y_1, \ldots, y_n$ at the points $x_1, \ldots, x_n$. Since there is a unique polynomial of degree at most $n-1$ that interpolates any set of n distinct points and n values, we conclude that both (1) and (2) have unique solutions.

Remark 1. In fact, these unique weights w_k are given by the explicit formula

$$w_k = \frac{2}{(1-x_k)^2 (P_n'(x_k))^2}$$

see for example page 887 of Abramowtiz and Stegun.

1.4. **Gaussian quadrature.** Let $x_1, \ldots, x_n$ be the roots of $P_n(x)$, and $w_1, \ldots, w_n$ be weights satisfying (1). Then, if $p(x)$ is a polynomail of degree at most $2n-1$ we have

$$\int_{-1}^{1} p(x)dx = \sum_{k=1}^{n} p(x_k) w_k. \tag{3}$$

Proof. First we observe that (3) holds when p is a polynomial of degree at most $n-1$, which follows from the possibility writing p as a linear combination of $P_0, \ldots, P_{n-1}$ and the linearity of integration. Next, by polynomial division we can write $p(x)$ as

$$p(x) = q(x)P_n(x) + r(x),$$

where q and r both have degree at most $n-1$. Since P_n is orthogonal to all polynomials of degree less than n we have

$$\int_{-1}^{1} p(x)dx = \int_{-1}^{1} r(x)dx = \sum_{k=1}^{n} r(x_k) w_k = \sum_{k=1}^{n} p(x_k) w_k,$$

where the final equality follows from the fact that

$$p(x_k) = q(x_k)P_n(x_k) + r(x_k) = r(x_k),$$

for all k since $x_1, \ldots, x_n$ are the roots of P_n. □

1.5. **Remainder formula.** Let $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ be as above. Suppose that f is a real-valued function on $[-1, 1]$ that has $2n$ continuous derivatives. Then,

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} f(x_k) w_k + \frac{2^{2n+1}(n!)^4}{(2n+1)((2n)!)^3} f^{(2n)}(\xi),$$

for some $0 < \xi < 1$, see for example page 887 of Abramowtiz and Stegun.

1.6. **Arbitrary intervals.** Let $x_1, \ldots, x_n$ be the roots of $P_n(x)$, and $w_1, \ldots, w_n$ be weights satisfying (1). If f is a real-valued function $[a, b]$ that has $2n$ continuous derivatives, then

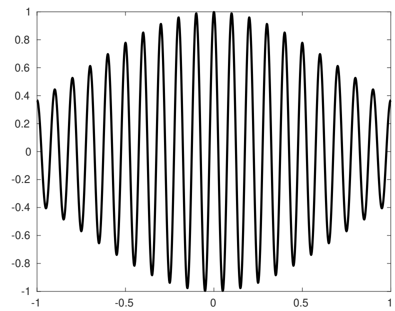
$$\int_{a}^{b} f(y)dy = \sum_{k=1}^{n} f\left(\frac{b-a}{2} x_k + \frac{b+a}{2}\right) w_k \frac{b-a}{2} + \frac{(2(b-a))^{2n+1}(n!)^4}{(2n+1)((2n)!)^3} f^{(2n)}(\xi),$$

where $a < \xi < b$. This follows from the standard remainder formula on $[-1, 1]$ by a change of variables.

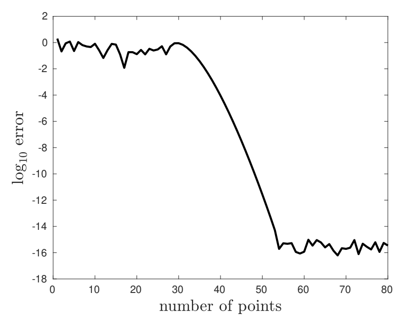
1.7. **Numerical example.** Informally speaking, the error in the Gaussian quadrature formula decreases super exponentially after we have "resolved the function" and the factorial term in the denominator of the remainder term becomes dominant. For example, consider the function

$$f(x) = \cos(20\pi x)e^{-x^2}$$

on the interval $[-1, 1]$.



Here is the error plot using Gaussian quadrature using an increasing number of points



After we are able to "resolve the oscillations" of the function f (which happens at approximately $n = 40$) the error decreases super exponentially to machine precision. If the function oscillates more in some places than others, then it may be necessary to use Gaussian quadrature adaptively to compute its integral accurately.